

# Decomposition of stochastic flows in manifolds with complementary distributions

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## Abstract

Let  $M$  be a differentiable manifold endowed locally with two complementary distributions, say horizontal and vertical. We consider the two subgroups of (local) diffeomorphisms of  $M$  generated by vector fields in each of these distributions. Given a stochastic flow  $\varphi_t$  of diffeomorphisms of  $M$ , in a neighbourhood of initial condition, up to a stopping time we decompose  $\varphi_t = \xi_t \circ \psi_t$  where the first component is a diffusion in the group of horizontal diffeomorphisms and the second component is a process in the group of vertical diffeomorphisms. Further decomposition will include more than two components; it leads to a maximal cascade decomposition in local coordinates where each component acts only in the corresponding coordinate.

**Key words:** stochastic flows, smooth distributions, decomposition of flows, group of diffeomorphisms.

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## 1 Introduction

Let  $M$  be a compact differentiable manifold, we shall consider  $\text{Diff}(M)$  the Lie group of smooth diffeomorphisms of  $M$  whose Lie algebra is given by smooth vector fields. Its Lie algebra is the usual bracket operation and the exponential map assigns to a vector field the unique flow that it generates (cf. Hamilton [7], Milnor [15]). Given a stochastic flow  $\varphi_t$  in  $\text{Diff}(M)$ , the decomposition of  $\varphi_t$  with components in subgroups of  $\text{Diff}(M)$  which provides dynamical or geometrical information of the system has been an interesting issue. The fact that one of the components is again a flow (or Markovian) turns a decomposition even more attractive in terms of applications. In the literature this kind of decomposition with different aimed subgroups has appeared among others in Bismut [1], Kunita [9], [10], Ming Liao [12] and some of our previous work [16], [3], [2]. In the last few papers mentioned, geometrical conditions on a Riemannian manifold have been stated to guarantee the existence of the decomposition where the first component lies in the subgroups of isometries or affine transformations.

In this article we consider the subgroups of  $\text{Diff}(M)$  whose elements preserve distributions in the sense of sections in a Grassmanian bundle of  $M$ . Given a pair of

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distribution, say a *horizontal*  $\Delta^H$  and a *vertical*  $\Delta^V$  distribution, we have associated to them the subgroups  $\text{Diff}(\Delta^H, M)$  and  $\text{Diff}(\Delta^V, M)$  generated by vector fields in the corresponding distribution. The main question addressed here is the possibility of decomposition of a stochastic flow as  $\varphi_t = \xi_t \circ \psi_t$ , with  $\xi_t \in \text{Diff}(\Delta^H, M)$  and  $\psi_t \in \text{Diff}(\Delta^V, M)$ .

One of the motivation for this decomposition appears in a foliated space, where one of the distributions is integrable. It corresponds to one of the possible answers to the following question: given trajectories of a dynamical systems in a foliated space which does not preserve foliation, how close is the system to be leaf-preserving? Yet in another words, how close is the vertical component to the identity? Transversal perturbation in the Liouville torus in Hamiltonian systems, cf. X.-M. Li [11], is an example in this context.

A principal bundle with a connection is another natural state space in this context, where the horizontal and vertical distributions are given by the geometry.

## 2 Main results

Let  $M$  be a compact connected  $n$ -dimensional differentiable manifold, here all geometric objects are considered smooth. Assume that (locally)  $M$  is endowed with a pair of regular differentiable distributions denoted by the *horizontal distribution*  $\Delta^H : U \subset M \rightarrow \text{Gr}_k(M)$ , and the *vertical distribution*  $\Delta^V : U \subset M \rightarrow \text{Gr}_{n-k}(M)$ , where  $U \subset M$  is a connected open set,  $\text{Gr}_k(M) = \cup_{x \in M} \text{Gr}_k(T_x M)$  is the Grassmannian bundle. We assume that the horizontal and the vertical distributions are complementary in the sense that  $\Delta^H(x) \oplus \Delta^V(x) = T_x M$ , for all  $x \in U$ .

We shall consider a stochastic flow  $\varphi_t$  generated by a Stratonovich SDE on  $M$ :

$$dx_t = \sum_{i=0}^m X_i(x_t) \circ dW_t^i, \quad (1)$$

where  $W_t^0 = t$ ,  $(W^1, \dots, W^m)$  is a Brownian motion in  $\mathbf{R}^m$  constructed on a filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$  and  $X_0, X_1, \dots, X_m$  are smooth vector fields in  $M$ . There exists a stochastic solution flow of (local) diffeomorphisms  $\varphi_t$ , see e.g. among others the classical Kunita [9], Elworthy [4].

In the Lie group  $\text{Diff}(M)$ , the dynamics of the stochastic flow  $\varphi_t$  is written as the right invariant equation:

$$d\varphi_t = \sum_{i=0}^m R_{\varphi_t*} X_i \circ dW_t^i, \quad (2)$$

where  $R_{\varphi_t*}$  is the derivative of the right translation in the group  $\text{Diff}(M)$ .

For a vector field  $X$  in  $M$ , the associated (local) flow is denoted by  $\exp\{tX\} \in \text{Diff}(M)$ . Given a distribution  $\Delta$  in  $M$ , we shall denote by  $\text{Diff}(\Delta, M)$  the group of diffeomorphisms generated by exponentials of vector fields in  $\Delta$ , precisely:

$$\text{Diff}(\Delta, M) = \text{cl}\{\exp\{t_1 X_1\} \circ \dots \circ \exp\{t_n X_n\}, \text{ with } X_i \in \Delta, t_i \in \mathbf{R}, \text{ for all } n \in \mathbf{N}\}.$$

With this notation, the stochastic flow  $\varphi_t \in \text{Diff}(TM, M)$ , a connected subgroup of  $\text{Diff}(M)$  which contains the Lie subgroups  $\text{Diff}(\Delta^H, M)$  and  $\text{Diff}(\Delta^V, M)$ .

If both distributions  $\text{Diff}(\Delta^H, M)$  and  $\text{Diff}(\Delta^V, M)$  are involutive, locally the intersection of these two subgroups is the identity, and the elements of each of these subgroups preserve the leaves of the corresponding foliation.

The main result of this section is the decomposition of the stochastic flows  $\varphi_t$  in  $M$  into components in  $\text{Diff}(\Delta^H, M)$  and  $\text{Diff}(\Delta^V, M)$ .

**Definition 2.1.** *We say that a pair of transversal complementary distributions  $\Delta^H$  and  $\Delta^V$  preserves transversality along the orbits of  $\text{Diff}(\Delta^H, M)$  acting on  $TM$  if  $\xi_*\Delta^V(\xi^{-1}(x)) \cap \Delta^H(x) = \{0\}$  for any element  $\xi$  in the group  $\text{Diff}(\Delta^H, M)$ .*

Equivalently,  $\Delta^H$  and  $\Delta^V$  preserves transversality if and only if  $(\xi_t)_*\Delta^V(\xi_t^{-1}(x)) \cap \Delta^H(x) = \{0\}$  along trajectories  $\xi_t$  of control or stochastic systems generated by horizontal vector fields. If  $\Delta^H$  is integrable then, for any complementary distribution  $\Delta^V$ , the pair  $\Delta^H$  and  $\Delta^V$  preserves transversality along  $\text{Diff}(\Delta^H, M)$ . In fact,  $\xi_*\Delta^H = \Delta^H \circ \xi$ , hence the property follows since  $\xi_*$  is an isomorphism.

Our technique consists on lifting Equation (1) to equations in the Lie subgroups  $\text{Diff}(\Delta^V, M)$  and  $\text{Diff}(\Delta^H, M)$ .

**Theorem 2.2** (Decomposition of flows. Global version). *Let  $\Delta^H$  and  $\Delta^V$  be two complementary distributions in  $M$  which preserves transversality along  $\text{Diff}(\Delta^H, M)$ . Given a stochastic flow  $\varphi_t$ , up to a stopping time there is a factorization*

$$\varphi_t = \xi_t \circ \psi_t$$

where  $\xi_t$  is a diffusion in  $\text{Diff}(\Delta^H, M)$  and  $\psi_t$  is a process in  $\text{Diff}(\Delta^V, M)$ .

**Remark 1.** The decomposition is local in the Lie group  $\text{Diff}(\Delta^H, M)$  and global in the manifold  $M$ . Compare to Remark 3.

*Proof.* Denote by  $\pi_{\Delta^V, \Delta^H} : T_x M \rightarrow \Delta^V \subset T_x M$  the projection onto the vertical distribution  $\Delta^V$  along  $\Delta^H$ . Define, for each  $i = 0, 1, \dots, m$ , and each element  $\xi$  in the Lie group  $\text{Diff}(\Delta^H, M)$ , the following Lie algebra element

$$\widetilde{X}_i(x) = X_i(x) - v_i(\xi, x) \quad (3)$$

where  $v_i(\xi, x)$  is the unique vector in the subspace  $\text{Ad}(\xi)\Delta^V \subset T_x M$  such that  $\widetilde{X}_i$  is horizontal, i.e.  $\pi_{\Delta^V, \Delta^H}(X_i(x) - v_i(x)) = 0$ .

Consider the following Stratonovich SDE in the subgroup  $\text{Diff}(\Delta^H, M)$  generated by the right action of  $\xi \in \text{Diff}(\Delta^H, M)$  in the Lie algebra elements ( $\xi$ -dependent)  $\widetilde{X}_i$ :

$$d\xi_t = \sum_{i=0}^m R_{\xi_t*} \widetilde{X}_i \circ dW_t^i, \quad (4)$$

with initial condition  $\xi_0 = Id$ . By the support theorem, the diffusion  $\xi_t$  lives in  $\text{Diff}(\Delta^H, M)$ , since  $\widetilde{X}_i$  are horizontal vector fields for all  $\xi$  and all  $x \in M$ . Classical existence results guarantee that there exists a solution of equation (4) up to a stopping time.

For the second component, we write  $\psi_t = \xi_t^{-1} \circ \phi_t$  and use that

$$d\xi_t^{-1} = - \sum_{i=0}^m L_{\xi_t^{-1}*} \widetilde{X}_i \circ dW_t^i.$$

Hence, by Itô formula:

$$\begin{aligned}
d\psi_t &= \sum_{i=0}^m \xi_t^{-1} X_i \xi_t \psi_t \circ dW_t^i - \xi_t^{-1} \tilde{X}_i \xi_t \psi_t \circ dW_t^i. \\
&= \sum_{i=0}^m \text{Ad}(\xi_t)(X_i - \tilde{X}_i) \psi_t \circ dW_t^i \\
&= \sum_{i=0}^m \text{Ad}(\xi_t)(v_i) \psi_t \circ dW_t^i.
\end{aligned}$$

Since, by construction,  $\text{Ad}(\xi_t)(v_i)(x) \in \Delta^V(x)$ , again by the support theorem, we have that  $\psi_t \in \text{Diff}(\Delta^V, M)$ .  $\square$

Although in general the factor  $\xi_t$  is not a solution of an autonomous SDE in the manifold itself, if the vertical distribution is invariant by  $\text{Ad}(\xi_t)$ , for  $t \geq 0$ , then  $\xi_t$  is indeed a solution of an SDE generated by (horizontal) vector fields in  $M$ :

**Corollary 2.3** (Horizontal diffusion on the manifold). *If  $\text{Ad}(\xi_t)\Delta^V = \Delta^V$  for  $t \geq 0$  then  $\xi_t$  is the solution of the following equation in  $M$ :*

$$dx_t = \sum_{i=0}^m X_i^H(x_t) \circ dW_t^i,$$

where  $X_i^H$  are the horizontal component  $\pi_{\Delta^H, \Delta^V} X_i$ , for each  $i = 0, 1, \dots, m$ .

*Proof.* Indeed, in this case, for each  $i = 0, 1, \dots, m$ , the vectors  $v_i$  in the proof of Theorem 2.2 is simply the vertical component  $X_i^V = \pi_{\Delta^V, \Delta^H} X_i$ .  $\square$

**Corollary 2.4** (Constant energy foliation). *Let  $M$  be a Riemannian manifold and  $h : M \rightarrow \mathbf{R}$  be a submersion. Then, up to a stopping time, the stochastic flow  $\varphi_t$  of equation (1) can be factorized as*

$$\phi_t = \xi_t \circ \psi_t$$

where the diffusion  $\xi_t$  preserves  $h$  and  $\psi_t$  is a process in the group of diffeomorphisms which acts orthogonally on the leaves of constant  $h$ .

*Proof.* Take  $\Delta^H = \text{Ker } h_*$  and  $\Delta^V = \{\lambda \nabla h; \lambda \in \mathbf{R}\}$ . These distributions are involutive and orthogonal. In this particular case, the vertical vectors  $v_i(x)$ ,  $i = 0, 1, \dots, m$  in the proof of Theorem 2.2 can easily be calculated at each point  $x \in U$  by:

$$v_i(x) = \frac{\langle \nabla h(x), X_i(x) \rangle}{\langle \nabla h(x), \xi_* \nabla h(\xi^{-1}(x)) \rangle} \xi_* \nabla h(\xi^{-1}(x)).$$

$\square$

Given a neighbourhood  $U_{x_0}$  of a point  $x_0 \in M$ , we shall denote by  $\text{Diff}(\Delta^H, U_{x_0}) = \{\varphi : U_{x_0} \rightarrow \varphi(U_{x_0})\}$  the set of diffeomorphisms of  $U_{x_0}$  generated by horizontal vector fields; analogously,  $\text{Diff}(\Delta^V, U_{x_0})$  denotes the diffeomorphisms generated by vertical vector fields.

Transversality condition along the orbit of  $\text{Diff}(\Delta^H, M)$  required in Theorem 2.2 can be suppressed in the local version.

**Theorem 2.5** (Decomposition of flows. Local version). *Assume that the manifold  $M$  is locally endowed with a pair of complementary distributions  $\Delta^H$  and  $\Delta^V$ . Given a stochastic flow of local diffeomorphisms  $\varphi_t$ , for each point  $x_0 \in M$  there exists an open neighbourhood  $U_{x_0} \subset M$  such that, up to a stopping time, we can decompose*

$$\varphi_t|_{U_{x_0}} = \xi_t \circ \psi_t$$

where  $\xi_t$  is a diffusion in  $\text{Diff}(\Delta^H, U_{x_0})$  and  $\psi_t$  is a process in  $\text{Diff}(\Delta^V, U_{x_0})$ .

*Proof.* The adjointly vertical correction term  $v_i(\xi, x)$  in equation (3) depends continuously on  $\xi$ . At the identity  $\xi_0 = Id$ , this equation gives  $v_i(\xi_0, x) = \pi_{\Delta^V, \Delta^H} X(x)$ , therefore it is well defined in a neighbourhood of  $\xi_0 = Id$  (in the space of diffeomorphisms), and in a neighbourhood of the initial condition  $x_0$  (in  $M$ ), where  $\Delta^H$  and  $\Delta^V$  are defined and complementary. So,  $v_i(\xi, \cdot)$  is defined up to a stopping time  $\tau$ , such that for  $t < \tau(\omega)$  at the point  $\varphi_t(\omega, x_0)$ ,  $\omega \in \Omega$ , the distributions  $\Delta^H$  and  $\Delta^V$  are defined with  $Ad(\xi_t)\Delta^V$  and  $\Delta^V$  complementary. Hence, the equations of the horizontal and vertical components  $\xi_t$  and  $\psi_t$  respectively, hold up to the minimum of the explosion of equation (4) and the stopping time  $\tau$ .  $\square$

**Corollary 2.6** (Involutive distributions). *If both  $\Delta^H$  and  $\Delta^V$  are involutive then the local decomposition is unique.*

*Proof.* Just note that in a neighbourhood  $U_{x_0} \subset M$  holonomy of the foliations vanish, i.e.  $\text{Diff}(\Delta^H, U_{x_0}) \cap \text{Diff}(\Delta^V, U_{x_0}) = Id$ .  $\square$

**Example 1:** Let  $M = \mathbf{R}^3$  with the canonical basis  $\{e_1, e_2, e_3\}$  and  $p = (x, y, z) \in M$ . Consider the distributions  $\Delta^H(p) = \text{span}\{(\cos y^2, 0, \sin y^2), (0, 1, 0)\}$  and  $\Delta^V(p) = \text{span}\{(-\sin y^2, 0, \cos y^2)\}$ . For the constant vector field  $Y \equiv e_2$  in  $\Delta^H(x)$ , the linearization of the corresponding flow  $\varphi_t \in \text{Diff}(\Delta^H, M)$  is the identity  $d\varphi_t = Id$ . There exists a sequence of points  $p_n$  and a sequence  $t_n \rightarrow 0$  such that transversality degenerates i.e.  $\varphi_{t_n}\Delta^V(\varphi_t^{-1}(p_n)) \subset \Delta^H(p_n)$ . Hence the pair of transversal distribution  $\Delta^H$  and  $\Delta^V$  are complementary but they do not preserve transversality globally along  $\text{Diff}(\Delta^H, M)$  according to Definition 2.1. Local decomposition as in Theorem 2.5 holds. Note that changing the order between the horizontal and the vertical distributions, we have that the pair of distributions does preserve transversality along  $\text{Diff}(\Delta^V, M)$  since  $\Delta^V$  is integrable.

**Example 2:** Coordinate systems in differentiable manifolds is a natural source of complementary involutive distributions. For instance, with the pair of foliation on  $\mathbf{R}^n - \{0\}$  given by spheres centred at the origin and the corresponding radial lines, the angular diffusion  $\xi_t$  projected in  $S^{n-1}$  and the radial component  $\psi_t$  of Theorem 2.2 is one of the examples related to Liao's factorization in [13] and [14].

**Example 3:** Let  $(\pi : P \rightarrow M, G)$  be a principal fibre bundle with a connection form  $\omega$ . It is convenient to define  $\Delta^H = \ker \pi_*$ , the tangent subspace of the orbits of the action of  $G$  and  $\Delta^V = \ker \omega$ , established by the geometry. In general a flow  $\varphi_t$  in manifold with two complementary distributions can not be decomposed into two diffusions in  $\text{Diff}(\Delta^H, M)$  and in  $\text{Diff}(\Delta^V, M)$  simultaneously. In the case of  $P$  being the frame bundle of the differentiable manifold  $M$  with structural group  $Gl(n, \mathbf{R})$

we have an interesting particular situation: For the stochastic flow  $\varphi_t$  in the base manifold  $M$ , consider the induced linearized flow  $\varphi_{t*}$  in  $P$ . It is well known that many informations for the dynamics of  $\varphi_t$  in  $M$  can be studied using the vertical and horizontal components of  $\varphi_{t*}$ , e.g. parallel transport, Lyapunov exponents, rotation number, and others (see e.g. among others [8], [5], [6] [16]). We can write (see e.g. [5])

$$\varphi_{t*}(u_0) = R_{g_t}(u_t^h),$$

where  $R_{g_t}$  is the right action of the structural group,  $g_t$  obtained by parallel transport and  $u_t^h \in P$  is the horizontal lift of an initial frame  $u_0$  with  $\pi(u_0) = x_0$ . For any continuous  $\gamma_t$  with  $\gamma_0 = Id$  in the group of holonomy of  $P$  we also have that

$$\varphi_{t*}(u_0) = R_{\gamma_t g_t}(u_t^h \gamma_t^{-1}),$$

which corresponds to one of the decompositions stated in Theorem 2.5. If the manifold has curvature zero then, by Corollary 2.6 we have uniqueness of decomposition, hence the horizontal component in  $\text{Diff}(\Delta^H, P)$  is  $\xi_t = R_{g_t}$  and  $\psi_t(u_0) = u_t^h$ . Moreover, the right action of the structural group in  $P$  preserves the connection  $\omega$ , hence Corollary 2.3 says that one can obtain an equation for  $\xi_t$  in  $P$  instead of in  $\text{Diff}(\Delta^H, P)$ , as it is expected by the geometry.

**Remark 2.** If in equations (2) and (4), instead of right translation  $R_{\varphi*}$  one considers the left translation  $L_{\varphi*}$ , one finds, left diffusions  $\varphi_t^L, \xi_t^L$  respectively and a vertical process  $\psi_t^L$  such that the decomposition stated in the theorems above changes the order:

$$\varphi_t^L = \psi_t^L \circ \xi_t^L.$$

□

### 3 Cascade Decomposition

In this section we assume that we have a sequence of complementary distributions. Precisely, consider two sequences of enclosing distributions  $\Delta_1^H \subset \dots \subset \Delta_k^H$  and  $\Delta_1^V \supset \dots \supset \Delta_k^V$ . This is equivalent of saying that  $(\Delta_1^H, \dots, \Delta_k^H)$  and  $(\Delta_k^V, \dots, \Delta_1^V)$  are smooth sections of a flag bundle over  $M$ . The sequence of distribution  $(\Delta_1^H, \dots, \Delta_k^H)$  is a section of the maximal flag manifold if  $\dim \Delta_{i+1}^H - \dim \Delta_i^H = 1$  and  $k = n$ .

We assume that for each  $i = 1, \dots, k \leq n$  the pair  $\Delta_i^H$  and  $\Delta_i^V$  are complementary, i.e.  $\Delta_i^H(x) \oplus \Delta_i^V(x) = T_x M$  for every  $x \in M$ . The enclosing hypothesis on the subspaces (i.e. they are sections of a flag bundle) induces an enclosing property in the corresponding generated Lie groups:  $\text{Diff}(\Delta_i^H, M) \subseteq \text{Diff}(\Delta_{i+1}^H, M)$  and  $\text{Diff}(\Delta_{i+1}^V, M) \subseteq \text{Diff}(\Delta_i^V, M)$  for each  $i = 1, \dots, k - 1$ .

**Theorem 3.1** (Cascade decomposition: Global version). *Let  $(\Delta_1^H, \Delta_2^H, \dots, \Delta_k^H)$  and  $(\Delta_k^V, \Delta_{k-1}^V, \dots, \Delta_1^V)$  be sequences of enclosing distributions (i.e. smooth sections of flag bundles) such that the pairs  $\Delta_i^H$  and  $\Delta_i^V$ ,  $i = 1, \dots, k$ , are complementary in tangent spaces and transversality is preserved along the action of  $\text{Diff}(\Delta_i^H, M)$ . Given an stochastic flow  $\varphi_t$  generated by equation (2), up to a stopping time we can decompose*

$$\varphi_t = \xi_t^1 \circ \dots \circ \xi_t^k \circ \Psi_t$$

where for each  $i = 1, \dots, k$ ,  $\xi_t^i \in \text{Diff}(\Delta_i^H, M)$ , the composition of the first  $i$ -th component  $(\xi_t^1 \circ \xi_t^2 \dots \circ \xi_t^i)$  is a diffusion in  $\text{Diff}(\Delta_i^H(M))$  and the composition of the last components  $(\xi_t^{i+1} \circ \dots \circ \xi_t^k \circ \Psi_t)$  is a process in  $\text{Diff}(\Delta_i^V; M)$ .  $\Psi_t = \text{Id}$  if  $\dim \Delta_k^H = n$ .

*Proof.* By Theorem 2.2, for each  $i = 1, \dots, k$  there exists a decomposition  $\varphi_t = \tilde{\xi}_t^{(i)} \circ \tilde{\Psi}_t^{(i)}$  such that  $\tilde{\xi}_t^{(i)}$  is a diffusion in  $\text{Diff}(\Delta_i^H, M)$  and  $\tilde{\Psi}_t^{(i)}$  lives in  $\text{Diff}(\Delta_i^V, M)$ . The result follows by taking  $\xi^1 = \tilde{\xi}_t^{(1)}$  and by induction

$$\xi_t^i = \left( \tilde{\xi}_t^{(i-1)} \right)^{-1} \circ \tilde{\xi}_t^{(i)}$$

for  $1 < i \leq k$  and  $\Psi_t = \tilde{\Psi}_t^{(k)}$ . If  $\dim \Delta_k^H = n$  then  $\text{Diff}(\Delta_k^V, M) = \{\text{Id}\}$ , which proves the last statement.  $\square$

**Corollary 3.2** (Cascade decomposition: local version). *Assume that  $M$  is locally endowed with pairs of complementary distributions  $\Delta_i^H$  and  $\Delta_i^V$ ,  $i = 1, 2, \dots, k$ , such that the sequences  $(\Delta_1^H, \Delta_2^H, \dots, \Delta_k^H)$  and  $(\Delta_k^V, \Delta_{k-1}^V, \dots, \Delta_1^V)$  are enclosed distributions (i.e. sections of flag bundles). Given the stochastic flow of local diffeomorphisms  $\varphi_t$  generated by equation (2), for each point  $x_0 \in M$  there exists an open neighbourhood  $U_{x_0} \subset M$  such that, up to a stopping time, we can decompose*

$$\varphi_t|_{U_{x_0}} = \xi_t^1 \circ \dots \circ \xi_t^k \circ \Psi_t$$

where for each  $i = 1, \dots, k$ ,  $\xi_t^i \in \text{Diff}(\Delta_i^H, U_{x_0})$ , the composition of the first  $i$ -th component  $(\xi_t^1 \circ \xi_t^2 \circ \dots \circ \xi_t^i)$  is a diffusion in  $\text{Diff}(\Delta_i^H, U_{x_0})$  and the composition of the last components  $(\xi_t^{i+1} \circ \dots \circ \xi_t^k \circ \Psi_t)$  is a process in  $\text{Diff}(\Delta_i^V; U_{x_0})$ .  $\Psi_t = \text{Id}$  if  $\dim \Delta_k^H = n$ .

*Proof.* By Theorem 2.5, for each  $i = 1, 2, \dots, k$  there exists a local decomposition  $\varphi_t = \tilde{\xi}_t^{(i)} \circ \tilde{\Psi}_t^{(i)}$  which holds up to a stopping time  $\tau_i$ . The result follows up to  $\tau = \min\{\tau_1, \tau_2, \dots, \tau_k\}$  repeating the construction of  $\xi_t^i$ , as in the proof of the Theorem 3.1.  $\square$

A particularly interesting situation is when the tangent space is decomposed as a direct sum of one-dimensional subspaces and we take maximal flag sections whose integrable distributions are generated by direct sum of these one-dimensional subspaces.

**Corollary 3.3** (Decomposition preserving local coordinates). *Let  $U \subset M$  be an open set with local coordinates  $\phi = (\phi_1, \dots, \phi_n) : U \subset M \rightarrow \mathbf{R}^n$ . Given a stochastic flow of local diffeomorphisms  $\varphi_t$  generated by equation (2), up to a stopping time, we can decompose locally*

$$\varphi_t = \xi_t^1 \circ \dots \circ \xi_t^n$$

where for each  $i = 1, \dots, n$ , the (local) diffeomorphism  $\xi_t^i$  preserves the  $j$ -th coordinates for all  $j \neq i$  (i.e.  $\phi_j(\xi_t^i(x))$  is constant for each  $x$  in the domain); the composition  $(\xi_t^1 \circ \xi_t^2 \circ \dots \circ \xi_t^i)$  is a diffusion of diffeomorphisms which preserves coordinates  $\phi_{i+1}, \phi_{i+2}, \dots, \phi_n$ .

*The decomposition, in this order, is unique.*

*Proof.* Define the sequence of complementary involutive distributions by the following: at each  $x \in U$ ,  $\Delta_i^H = \text{span}\{\nabla\phi_1, \dots, \nabla\phi_i\}$  and  $\Delta_i^V = \text{span}\{\nabla\phi_{i+1}, \dots, \nabla\phi_n\}$ ,  $i = 1, \dots, n$ . Diffeomorphisms in  $\text{Diff}(\Delta_i^H, M)$  preserve the leaves of the foliation induced by  $\Delta_i^H$ , i.e. it preserves coordinates  $\phi_j$  if  $j > i$ . Analogously, diffeomorphisms in  $\text{Diff}(\Delta_i^V, M)$  preserve coordinates  $\phi_j$  if  $j \leq i$ .

Corollary 3.2 guarantees that

$$\varphi_t = \xi_t^1 \circ \dots \circ \xi_t^n$$

with  $(\xi_t^1 \circ \dots \circ \xi_t^i) \in \text{Diff}(\Delta_i^H, M)$  for  $i = 1, \dots, n$ . Hence, since both  $(\xi_t^1 \circ \dots \circ \xi_t^i)$  and  $(\xi_t^1 \circ \dots \circ \xi_t^{i-1})$  preserve coordinates  $\phi_j$  for  $j > i$ , then  $\xi_t^i$  also preserves coordinates  $\phi_j$  for  $j > i$ .

Moreover,  $(\xi_t^i \circ \xi_t^{i+1} \circ \dots \circ \xi_t^n) \in \text{Diff}(\Delta_{i-1}^V, M)$ . Hence, since both  $(\xi_t^{i+1} \circ \dots \circ \xi_t^n)$  and  $(\xi_t^i \circ \dots \circ \xi_t^n)$  preserve coordinates  $\phi_j$  for  $j \leq i-1$ , then  $\xi_t^i$  also preserves coordinates  $\phi_j$  for  $j \leq i-1$ .  $\square$

Next Lemma gives the picture of the restriction for existence of the decomposition we treat in this article. This restriction appears as an explosion time in the SDE for the components of the decomposition, cf. equation (4).

Given a diffeomorphism  $\varphi : U \rightarrow V$  between open sets  $U, V \in \mathbf{R}^n$ , with  $x = (x_1, \dots, x_n) \mapsto (\varphi_1(x), \dots, \varphi_n(x))$ . We denote the lower right  $(n-i+1) \times (n-i+1)$ -submatrix of the differential  $\varphi_*$  with respect to the canonical basis by

$$\frac{\partial(\varphi_i, \dots, \varphi_n)}{\partial(x_i, \dots, x_n)} := \left[ \frac{\partial\varphi_j}{\partial x_k} \right]_{i \leq j \leq n, i \leq k \leq n}.$$

Let  $\text{Dif}^i(U)$  be the set of diffeomorphisms  $\xi : U \rightarrow \xi(U)$  which only acts on the  $i$ -th coordinate, i.e.  $\xi(x) = (x_1, \dots, x_{i-1}, \xi_i(x), x_{i+1}, \dots, x_n)$ .

**Lemma 3.4.** *Let  $U \subset \mathbf{R}^n$  be an open set.*

(i) *A diffeomorphism  $\varphi : U \rightarrow V$  can be written as a composition of a sequence of diffeomorphisms*

$$\varphi = \xi^1 \circ \dots \circ \xi^n \tag{5}$$

*where each  $\xi^i \in \text{Dif}^i(U_x)$  changes only the  $i$ -th coordinate in an open neighbourhood  $U_x \subset U$  of  $x$  if and only if  $\det \frac{\partial(\varphi_i, \dots, \varphi_n)}{\partial(x_i, \dots, x_n)}(x) \neq 0$  for all  $1 \leq i \leq n$ .*

(ii) *The decomposition of equation (5), in this order, is unique.*

(iii) *If for a certain  $1 \leq i \leq n$ ,  $\det \frac{\partial(\varphi_i, \dots, \varphi_n)}{\partial(x_i, \dots, x_n)}(x)$  goes to zero as  $x \rightarrow p \in \text{cl}(U)$  then  $\|\xi^i\|_{C^1}$  increases to infinity in  $U_x$ .*

(iv) *The set of diffeomorphisms  $\varphi \in \text{Dif}(U)$  which admits the decomposition above is a dense open set containing the identity in the space of diffeomorphisms  $\text{Dif}(U)$  with respect to the  $\|\cdot\|_{C^1}$  topology.*

*Proof.* (i) Write in local coordinates  $\varphi = (\varphi_1, \dots, \varphi_n)$ . The maps

$$x \mapsto (x_1, \dots, x_k, \varphi_{k+1}(x), \dots, \varphi_n(x))$$

are diffeomorphisms in a neighbourhood of  $x \in U$ . Define, for each  $1 \leq k \leq n$  the diffeomorphisms

$$\xi^k = (x_1, \dots, x_{k-1}, \varphi_k(x), \dots, \varphi_n(x)) \circ (x_1, \dots, x_k, \varphi_{k+1}(x), \dots, \varphi_n(x))^{-1}. \tag{6}$$



They satisfy the required condition. The converse is trivial.

(ii) Uniqueness follows because the intersections of the sets  $\text{Dif}^i(U_a)$  and  $\text{Dif}^j(U_a)$  is the identity if  $i \neq j$ .

(iii) It follows by formula (6).

(iv) The required condition for existence of decomposition  $\det \frac{\partial(\varphi_i, \dots, \varphi_n)}{\partial(x_i, \dots, x_n)}(a) \neq 0$  for all  $1 \leq i \leq n$  is satisfied in an open dense subset of  $\text{Dif}(U)$ .  $\square$

**Remark 3.** The statement of the lemma above has considered the maximal flag manifold in order to provide the total cascade decomposition for flows in  $\mathbf{R}^n$ . The result can easily be restated in terms of just a pair of complementary distribution of any dimension, as in the hypothesis of the previous section. In the formulae for the components of the decomposition of a flow  $\varphi_t = \xi_t \circ \psi_t$  as in Theorem 2.2, the equation of  $\xi_t$  explodes when  $\varphi_t$  reaches points outside the dense open set containing the identity, as stated in the item (iv) of Lemma 3.4. An illustrative example is the linear planar systems  $(x', y') = (-y, x)$ , where the rotation  $\varphi_t$  hits a point outside the dense open set. In fact, easy calculation of the components of  $\varphi_t = \xi_t \circ \psi_t$  shows that the entry  $[\xi_t]_{1,2}$  satisfies

$$\frac{d}{dt}[\xi_t]_{1,2} = -1 - [\xi_t]_{1,2}^2,$$

which explodes at  $t = \pi/2$ .

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